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Topology and its Applications 122 (2002) 425–451

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Independent group topologies on Abelian groups ☆

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Received 27 October 1999; received in revised form 13 May 2000

Abstract

Two non-discrete T_1 topologies τ_1, τ_2 on a set X are called *independent* if their intersection $\tau_1 \cap \tau_2$ is the cofinite topology on X. We show that a countable group does not admit a pair of independent group topologies. We use MA to construct group topologies on the additive groups $\mathbb{R}$ and $\mathbb{T}$ independent of their usual interval topologies. These topologies have necessarily to be countably compact and cannot contain convergent sequences other than trivial. It is also proved that all proper unconditionally closed subsets of an Abelian (almost) torsion-free group are finite. Finally, we generalize the result proved for $\mathbb{R}$ and $\mathbb{T}$ by showing that every second countable group topology on an Abelian group of size 2^ω without non-trivial unconditionally closed subsets admits an independent group topology (this also requires MA). In particular, this implies that under MA, every (almost) torsion-free Abelian group of size 2^ω admits a Hausdorff countably compact group topology.

AMS classification: Primary 22A05; 54H11, Secondary 54A25; 54A35; 54G20

Keywords: Independent topology; Transversal topology; Dual group; Countably compact; Convergent sequence; Unconditionally closed set; Martin's Axiom; Continuum Hypothesis

1. Introduction

All topologies we consider are assumed to be T_1. Therefore, we only deal with Hausdorff topological groups.

Two non-discrete topologies τ_1, τ_2 on a set X are called T_1-*complementary* if their union $\tau_1 \cup \tau_2$ generates the discrete topology and their intersection $\tau_1 \cap \tau_2$ is the cofinite topology

☆ Work partially supported by Mexican National Council of Sciences and Technology (CONACyT), Grant Number 400200-5-28411E.

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PII: S0166-8641(01)00161-4

on X, see [1,2,33–36,39]. The notion of T_1-complementarity is naturally split into two components: *transversality* ($\tau_1 \cup \tau_2$ generates the discrete topology) and *independence* ($\tau_1 \cap \tau_2$ is the cofinite topology).

The study of transversal and independent group topologies on groups was initiated in [14]. It was shown there that there are no T_1-complementary group topologies on infinite groups, so our division of T_1-complementarity to transversality and independence is especially natural in topological groups. The main results of [14] concern topological groups that admit transversal group topologies. This class contains many (but not all) locally compact groups, including the additive group of the reals $\mathbb{R}$, the multiplicative group of the complex numbers $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the general linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$, etc. Here we study the class of topological groups that admit an independent group topology.

Surprisingly, the investigation of independent group topologies immediately involves two seemingly unrelated topics: *unconditionally closed subsets* of topological groups and countably compact group topologies without non-trivial convergent sequences. A subset A of an abstract group G is called *unconditionally closed* in G if A is closed in $(G, \mathcal{T})$ for every Hausdorff group topology $\mathcal{T}$ on G [28]. Clearly, a group with a proper infinite unconditionally closed subset does not admit two independent Hausdorff group topologies, so a necessary condition for a group to admit an independent group topology is that it does not have such a subset. Another necessary condition appears in Theorem 2.3: countable groups do not have independent Hausdorff group topologies. These facts suggest the conjecture that every uncountable Abelian group without proper infinite unconditionally closed subsets admits a couple of independent Hausdorff group topologies (see Problem 6.4). Corollary 5.4, which in a sense summarizes almost all main results of the article, confirms this for Abelian groups of size 2^ω, but our proof of this fact depends on Martin's Axiom (MA for short). However, we have no examples of independent group topologies in ZFC. Proposition 2.4 explains the difficulty in constructing independent topologies: if one of two independent topologies τ_1, τ_2 is sequential, then the other has to be countably compact and cannot contain convergent sequences other than trivial.

The existence of infinite countably compact topological groups without non-trivial convergent sequences is an old open problem [15,4]. Several examples of such groups were constructed under additional set-theoretic assumptions like CH or MA (see [15,27, 38]). Since the group $\mathbb{R}$ with the usual topology τ is second countable (hence sequential), any construction of a group topology $\mathcal{T}$ on $\mathbb{R}$ independent of τ will "necessarily" involve the use of MA or CH. Actually, we prove in Theorem 3.1 that MA implies the existence of an independent group topology on many second countable topological groups, including $\mathbb{R}$, $\mathbb{T}$, etc. Any construction of an infinite countably compact topological group without non-trivial convergent sequences is easily converted into a construction of two countably compact topological groups whose product is not countably compact [15]. Several examples of a countably compact topological group whose square fails to be countably compact were constructed under MA in [21,41,42]. However, the problem of whether such a group exists in ZFC is still open (see [15], [4], [6, Section 5]).

In 1955, A.D. Wallace (see [44]) posed the following problem: Is it true that every cancellative countably compact topological semigroup is a topological group? Again, countably compact groups without non-trivial convergent sequences play an important role in the solution of Wallace's problem given by Robbie and Svetlichny in [30]. They showed that there exists a countably compact subsemigroup H of the countably compact topological group G constructed in [38] that does not contain the identity of G. This answers Wallace's question in the negative. However, the construction of the group G in [38] requires CH, so Robbie–Svetlichny's solution also depends on CH. Another counterexample to the Wallace problem was constructed under MA by Tomita in [40].

Our study of independent group topologies has yet another ingredient. The existence of a compact group topology on an abstract Abelian group G implies strong restrictions on the algebraic structure of G. Actually, the algebraic structure of compact Abelian groups is completely described by Kaplansky, Harrison and Hulanicky, see [22, 25.25]. The study of cardinal restrictions on the groups that admit a pseudocompact group topology was initiated by van Douwen [16]. The simplest one is that the cardinality of an infinite pseudocompact group is at least 2^ω. Then Comfort and Robertson showed in [7] that every torsion Abelian pseudocompact group G has a finite order, i.e., there exists a positive integer n such that $nx = e_G$ for all $x \in G$. This work was continued by Dikranjan and Shakhmatov in [10] where they found several new algebraic constraints for pseudocompact Abelian topological groups. In general, the existence of a pseudocompact group topology on a group G depends on extra set-theoretic axioms, but in the case $|G| = 2^\omega$ it does not. According to [10], an Abelian group G of size 2^ω admits a pseudocompact Hausdorff group topology iff either $r_0(G) = 2^\omega$ (equivalently, $|G/tor(G)| = 2^\omega$, where $tor(G)$ is the torsion subgroup of G) or G has a finite order n and $mG = \{mx\colon x \in G\}$ is either finite or has the size 2^ω for every divisor m of n. It is natural, therefore, to restrict ourselves to considering the second countable Abelian topological groups that admit an independent group topology. Since "independent" group topologies on the groups from this class are countably compact by Proposition 2.4, we have to expect these groups to satisfy even stronger algebraic and cardinal restrictions. Two of them have been mentioned: all proper unconditionally closed subsets of these groups are finite, and every infinite group of this class has the cardinality 2^ω. It turns out that no more restrictions exist (at least under MA). In fact, the first one can be given a more transparent form. Let us say that an Abelian group G is *almost torsion-free* if the subgroup $G[n] = \{x \in G\colon nx = 0_G\}$ is finite for each positive integer n. We characterize the Abelian groups without non-trivial unconditionally closed subsets in Theorem 5.1: they are either almost torsion-free or have a prime order p. This characterization is used to obtain the main result of the paper: under MA, every second countable Abelian topological group of size 2^ω without non-trivial unconditionally closed subsets admits an independent group topology (see Theorem 5.2). This implies, in particular, that every Abelian almost torsion-free group of size 2^ω admits a countably compact Hausdorff group topology. The latter result introduces the first class of such groups which is not contained in the class of groups described in the Kaplansky–Harrison–Hulanicky theorem [22, 25.25]. A complete characterization of “small” Abelian groups that admit a countably compact Hausdorff group topology will appear in the forthcoming paper [12]. It is based on techniques that we develop here.

The article is composed as follows. In Section 2, we show that countable groups do not admit independent group topologies. Our proof of this fact is based on (the special case of) Arhangel’skiĭ’s theorem that every group topology on a countable group contains a weaker group topology with a countable base. Then we prove Proposition 2.4 according to which a Hausdorff topology independent of a sequential topology has necessarily to be countably compact and to make all infinite sequences divergent.

In Section 3, we consider Abelian almost torsion-free groups (a.t.-f. for short). Clearly, the class of a.t.-f. groups contains all torsion-free groups as well as the circle group $\mathbb{T}$, all finite powers of $\mathbb{T}$, etc. The reader familiar with the theory of Abelian groups can immediately note that a group G is a.t.-f. iff the p-rank $r_p(G)$ of G (i.e., the cardinality of a maximal independent family of elements of order p) is finite for each prime p. Then we show in Theorem 3.1 that under MA, every second countable Abelian a.t.-f. topological group G of size 2^ω admits an independent group topology which is necessarily countably compact and makes all infinite sequences divergent. In particular, the groups $\mathbb{R}, \mathbb{T}, \mathbb{C}^*$ (endowed with their usual topologies) admit an independent group topology. We also prove in Theorem 3.13 that all proper unconditionally closed subsets of an Abelian a.t.-f. group are finite. This result does not require any extra set-theoretic assumptions and shows a way for possible generalizations of Theorem 3.1.

Abelian groups of a prime order are considered in Section 4. First, we prove an analog of Theorem 3.1: under MA, every second countable Abelian topological group G of a prime order with $|G| = 2^\omega$ admits an independent group topology (see Theorem 4.3). Then we show in ZFC that an Abelian group G of a prime order does not contain non-trivial unconditionally closed subsets no matter how big G is (Proposition 4.6).

In Section 5, we join the results of Sections 3 and 4. The Abelian groups without non-trivial unconditionally closed subsets are characterized in Theorem 5.1: they are either almost torsion-free or have a prime order. Combining Theorems 3.1, 4.3 and 5.1, we deduce Corollary 5.4: under MA, every Abelian group of size 2^ω without non-trivial unconditionally closed subsets admits a couple of independent group topologies.

Finally, we will discuss the technique employed here. The notion of *HFD* subsets of uncountable products introduced by Hajnal and Juhász [19] in the general topological context plays an important role in the article. The technique of *HFD* subsets also proved to be useful for constructions of “bizarre” topological groups [20,38]. In fact, we adjust this method to topological groups by proving Lemmas 3.4 and 4.2 for almost torsion-free groups and groups of a prime order, respectively. Without any exaggeration, these lemmas form the core of the paper: all main results here depend on them in some form. Roughly speaking, Lemma 3.4(a) means that for any countable family $\{S_n\colon n \in \omega\}$ of infinite subsets of an Abelian a.t.-f. group G, the set H of all homomorphisms $f : G \to \mathbb{T}$ that send each S_n to a dense subset of $\mathbb{T}$ is very large: the complement $G^* \setminus H$ is of the first category in the compact dual group G^*. Lemma 4.2(a) has a similar sense and applies to Abelian groups of a prime order p, so the group $\mathbb{T}$ has to be replaced by $\mathbb{Z}(p)^\omega$. Our proofs of Lemmas 3.4 and 4.2 are relatively simple, they only use the definition of Pontryagin's compact-open topology on the dual group G^* which coincides with the pointwise convergence topology when G is discrete. It is worth mentioning that this couple of lemmas implies a number of results proved earlier by distinct methods, somewhat complicated on occasion (see [20, Lemma 2.1], [38, Lemma 2], [11, Lemma 5.2]).

We use Martin's Axiom in the proofs of all results concerning the existence of independent group topologies. This use, however, is very simple. First, we apply the topological form of Martin's Axiom given in [25]: a compact Hausdorff space X of countable cellularity cannot be represented as a union of less than 2^ω nowhere dense subsets. Since the Pontryagin dual group G^* of a discrete Abelian group G is compact, hence dyadic [26,43], the cellularity of G^* is countable. This observation makes our applications of MA in Lemmas 3.4(b) and 4.2(b) fairly transparent. The appearance of MA in other results is particularly due to applications of these lemmas.

1.1. Notation and terminology

We recall here some compactness-like conditions on a topological group G. A group G is *precompact* if it can be covered by finitely many translates of any neighborhood of identity, *pseudocompact* if every continuous real-valued function on G is bounded, *countably compact* if every infinite subset of G has a cluster point in G, and *ω-bounded* if the closure of every countable subset of G is compact. Clearly, ω-bounded groups are countably compact, countable compactness implies pseudocompactness, and pseudocompact groups are precompact [8].

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$.

We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive natural numbers and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. The complex plane is $\mathbb{C}$, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the multiplicative group of non-zero complex numbers. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$ and for every $p \in \mathbb{P}$, $\mathbb{Z}_{p^\infty}$ is the quasicyclic subgroup of $\mathbb{T}$.

We use the multiplicative notation in the non-Abelian case, while the plus sign denotes the group operation in Abelian groups. The symbols e_G and 0_G (or simply e and 0) stand for the neutral element of a group G in the general and Abelian cases, respectively.

Let G be an Abelian group. The *torsion part* $t(G)$ of G is defined to be the set $\{g \in G\colon ng = 0 \text{ for some } n \in \mathbb{N}\}$. Clearly, $t(G)$ is a subgroup of G. The group G is *divisible* if the equation $nx = g$ has a solution $x \in G$ for all $g \in G$ and $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $G[n] = \{g \in G\colon ng = 0\}$. If p is prime, the *p-rank* of G, $r_p(G)$, is defined as the cardinality of a maximal independent subset of $G[p]$ (see [31, Section 4.2]). We say that G is *almost torsion-free* if $r_p(G)$ is finite for each $p \in \mathbb{P}$.

The cardinality of the continuum 2^ω will be also denoted by $\mathfrak{c}$. The weight of a space X is $w(X)$.

2. Transversal and independent topologies

We start with two definitions; the first of them introduces the main objective of our study: *independent topologies*. Recall that we only consider T_1 topologies.

Definition 2.1. Two topologies τ_1 and τ_2 on a set X are said to be *independent* if their intersection $\tau_1 \cap \tau_2$ is the cofinite topology on X.

Definition 2.2. Two non-discrete topologies τ_1 and τ_2 on a set X are called *transversal* if the union $\tau_1 \cup \tau_2$ generates the discrete topology on X.

Two group topologies τ_1, τ_2 on an infinite group G cannot be simultaneously transversal and independent by Theorem 2.7 of [14]. Therefore, if the topologies τ_1 and τ_2 are independent, their union $\tau_1 \cup \tau_2$ generates a non-discrete group topology on G. Many well-known topological groups (including $\mathbb{Q}$, $\mathbb{R}$, $GL(n, \mathbb{R})$, etc.) admit a transversal group topology. However, infinite precompact groups do not have a transversal group topology [14]. Some Abelian groups are *extremally transversable*: for example, every non-precompact group topology on $\mathbb{Z}$ admits a transversal one [14]. The case of independent group topologies is very different. Let us show that no countable topological group admits an independent group topology.

Theorem 2.3. *A countably infinite group does not admit independent Hausdorff group topologies.*

Proof. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are independent group topologies on a countably infinite group G. By Arhangel'skii's result in [3], for every $i = 1, 2$, there exists a Hausdorff group topology $\tau_i \subseteq \mathcal{T}_i$ on G such that $w(G, \tau_i) = \omega$. Clearly, the topologies τ_1 and τ_2 are also independent. From Theorem 2.7 of [14] it follows that the union $\tau_1 \cup \tau_2$ generates a non-discrete group topology τ on G. Since $w(G, \tau) = \omega$, there exists a non-trivial sequence $\{x_n \colon n \in \omega\}$ in (G, τ) converging to the identity e of G. Then $K = \{e\} \cup \{x_n \colon n \in \omega\}$ is an infinite closed subset of (G, τ_1) and (G, τ_2) with the infinite complement $G \setminus K$. This contradicts the independence of τ_1 and τ_2. $\square$

It turns out, however, that the situation changes in the case of uncountable groups: according to Corollary 3.7, each of the additive groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{C}$ admits a Hausdorff group topology independent of its usual Euclidean topology (this requires MA). Any construction of an independent group topology on these groups has necessarily to be quite complicated because of the following simple observation.

Proposition 2.4. *Suppose that τ and $\mathcal{T}$ are independent Hausdorff topologies on a set X. If the topology τ is sequential, then the space $(X, \mathcal{T})$ is countably compact and does not contain non-trivial convergent sequences.*

Proof. First, let us show that the space $(X, \mathcal{T})$ is countably compact. Suppose, to the contrary, that A is an infinite closed discrete subset of $(X, \mathcal{T})$. Clearly, $|X \setminus A| \geqslant \omega$, so A cannot be closed in (X, τ). Since (X, τ) is sequential, there exists a sequence $B \subseteq A$ converging to a point $a \in X \setminus A$. Then the set $B \cup \{a\}$ is closed in both spaces $(X, \mathcal{T})$ and (X, τ), a contradiction.

Finally, suppose that A is a non-trivial sequence in $(X, \mathcal{T})$ converging to a point $a \in X$. Choose an infinite subset $B \subseteq A$ such that $|A \setminus B| = \omega$. Since $|X \setminus B| \geqslant \omega$, the set $B \cup \{a\}$ cannot be closed in (X, τ). Therefore, there exists a non-trivial sequence $C \subseteq B$ converging to a point $b \in X$ in (X, τ). Clearly, the proper infinite subset $C \cup \{a, b\}$ of X is closed in (X, τ) and $(X, \mathcal{T})$, a contradiction. □

By [22, Example 25.26(c)], the abstract group $\mathbb{R}$ admits a compact Hausdorff group topology. However, no such topology is independent of the usual topology τ on $\mathbb{R}$. Indeed, every infinite compact topological group is dyadic [24,26], and hence contains non-trivial convergent sequences [17,18]. Therefore, a compact Hausdorff group topology on $\mathbb{R}$ cannot be independent of τ by Proposition 2.4. We shall show in the next section that under MA, there exists a countably compact, connected and locally connected group topology on $\mathbb{R}$ independent of τ (see Corollary 3.7).

3. Almost torsion-free groups

Recall that an Abelian group G is *almost torsion-free* if $G[n] = \{x \in G\colon nx = 0\}$ is finite for each $n \in \mathbb{N}$. Clearly, the circle group $\mathbb{T}$ is almost torsion-free. There are infinite torsion groups that are almost torsion-free, for example, the quasicyclic group $\mathbb{Z}_{p^\infty}$ for a prime p, or the subgroup $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$. It is easy to see that all such groups are countable. Actually, the torsion subgroup of an almost torsion-free Abelian group is countable.

Now we present the first main result of the paper.

Theorem 3.1. *Suppose that* MA *holds. Then every Abelian almost torsion-free topological group (G, τ) satisfying $|G| = \mathfrak{c} = |\tau|$ admits a Hausdorff group topology $\mathcal{T}$ with the following properties*:

(a) *the topologies τ and $\mathcal{T}$ are independent*;
(b) *the group $(G, \mathcal{T})$ is countably compact and does not contain non-trivial convergent sequences*;
(c) *the group $(G, \mathcal{T})$ is connected and locally connected.*

The following lemma gives a basic idea of our construction of independent group topologies.

Lemma 3.2. *Let K be a compact topological group, (G, τ) be a topological group of cardinality $\mathfrak{c}$, and suppose that $h : G \to K^{\mathfrak{c}}$ is an algebraic monomorphism satisfying*

(1) *$h(G)$ is a countably compact subgroup of $K^{\mathfrak{c}}$ without non-trivial convergent sequences*;

(2) *$h(F)$ is dense in $K^{\mathfrak{c}}$ for every closed subset F of (G,τ) with $|F| = \mathfrak{c}$.*

Denote by $\mathcal{T}$ the topology on G generated by h, i.e., $\mathcal{T} = \{h^{-1}(U \cap h(G))\colon U$ is open in $K^{\mathfrak{c}}\}$. Then the topologies $\mathcal{T}$ and τ are independent.

Proof. Take a non-empty set $V \in \tau$. If $|G \setminus V| = \mathfrak{c}$, then $h(G \setminus V)$ is dense in $h(G)$ by (2). This implies that $h(V)$ is not open in $h(G)$, so $V \notin \mathcal{T}$. If $\omega \leqslant |G \setminus V| < \mathfrak{c}$, then $h(G \setminus V)$ cannot be closed in $h(G)$. Indeed, otherwise the image $h(G \setminus V)$ would be countably compact, hence scattered. However, every infinite, regular, countably compact, scattered space contains a non-trivial convergent sequence, thus contradicting (1). In other words, $h(V)$ is not open in $h(G)$ either, and hence $V \notin \mathcal{T}$. $\square$

To construct an appropriate monomorphism $h : G \to K^{\mathfrak{c}}$ satisfying (1) and (2) of the previous lemma, we need three auxiliary results. The first of them generalizes and strengthens Lemma 5.2 of [11] (proved in the special case $G = \mathbb{Z}$) and is the main technical tool in the article. Its proof involves the use of Pontryagin's *dual group* G^* for an Abelian group G.

Let G be a discrete Abelian group. Denote by G^* the group of characters of G, i.e., the set of all homomorphisms $f : G \to \mathbb{T}$ with the sum operation $(f+g)(x) = f(x) + g(x)$, where $x \in \mathbb{T}$ and $f, g \in G^*$. The constant homomorphism 0 is the identity of G^*. For $f \in G^*$, $x_1, \ldots, x_m \in G$ and $\varepsilon > 0$, put

$$W(f, x_1, \ldots, x_m, \varepsilon) = \bigl\{h \in G^*\colon \bigl|h(x_i) - f(x_i)\bigr| < \varepsilon \text{ for each } i = 1, \ldots, m\bigr\}.$$

It is known (see [29,22,9]) that the family

$$\bigl\{W(f, x_1, \ldots, x_m, \varepsilon)\colon f \in G^*,\ x_1, \ldots, x_m \in G,\ \varepsilon > 0\bigr\}$$

forms a base of a compact Hausdorff group topology on G^*.

Since the group $\mathbb{T}$ is divisible, every homomorphism $f : H \to \mathbb{T}$ defined on a subgroup H of G extends to a homomorphism $h : G \to \mathbb{T}$ [31]. Actually, for every $x \in G \setminus H$ and $t \in \mathbb{T}$, there exists a homomorphism $h : G \to \mathbb{T}$ extending f and satisfying $h(x) = t$ provided that x and t satisfy $k \cdot t = f(kx)$, where $\min\{n \in \mathbb{N}\colon nx \in H\} = k < \infty$. This is also the case if $\langle x\rangle$ is an infinite cyclic group and $\langle x\rangle \cap H = \{0\}$. These facts will be implicitly used in the proofs of several results, including the following key lemma and Theorem 3.1.

Lemma 3.3. *Let S be an infinite subset of an Abelian group G such that the intersection $S \cap (x + G[n])$ is finite for all $x \in G$ and $n \in \mathbb{N}$. Then the set*

$$H_S = \bigl\{h \in G^*\colon h(S) \text{ is dense in } \mathbb{T}\bigr\}$$

is an intersection of countably many open dense subsets of G^*.

Proof. Let $\{s_n\colon n \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{T}$. For every $n \in \mathbb{N}$, consider the set

$$U_n = \bigl\{h \in G^*\colon \exists x \in S \text{ such that } \bigl|h(x) - s_n\bigr| < 1/n\bigr\}.$$

Clearly, the sets U_n are open in G^*. Let us show that $H_S = \bigcap_{n=1}^{\infty} U_n$ and each U_n is dense in G^*.

We start with the inclusion $P = \bigcap_{n=1}^{\infty} U_n \subseteq H_S$. It suffices to verify that $h(S)$ is dense in $\mathbb{T}$ for each $h \in P$. Indeed, if $h \in P$ and O is a non-empty open set in $\mathbb{T}$, choose $n \in \mathbb{N}$ such that $s_n \in O$ and the open interval with the center at s_n and of radius $1/n$ is contained in O. From $h \in U_n$ it follows that $|h(x) - s_n| < 1/n$ for some $x \in S$, whence $h(S) \cap O \neq \emptyset$. This proves the density of $h(S)$ in $\mathbb{T}$, so $P \subseteq H_S$. The inverse inclusion $H_S \subseteq P$ is immediate.

It remains to show that every U_n is dense in G^*. Let $n \in \mathbb{N}$ be arbitrary. Take a basic open set $W_0 = W(f, z_1, \ldots, z_r, \varepsilon)$ in G^*, where $f \in G^*$, $z_1, \ldots, z_r \in G$ and $\varepsilon > 0$. We claim that $W_0 \cap U_n \neq \emptyset$.

Denote by H the subgroup of G generated by the elements $z_1, \ldots, z_r$. Since H is a finitely generated Abelian group, there are $y_1, \ldots, y_m \in H$ such that $H = \bigoplus_{i=1}^{m} \langle y_i \rangle$ [31, 4.2.10]. Note that $m \leqslant r$. The elements $z_1, \ldots, z_r$ are linear combinations of $y_1, \ldots, y_m$, so we can find $\delta > 0$ such that $W_1 = W(f, y_1, \ldots, y_m, \delta) \subseteq W(f, z_1, \ldots, z_r, \varepsilon)$. Therefore, it suffices to show that $W_1 \cap U_n \neq \emptyset$.

For every $x \in S$, define $d(x) = \min\{k \in \mathbb{N}\colon kx \in H\}$ (if $kx \notin H$ for every $k \in \mathbb{N}$, we put $d(x) = \infty$). Let us consider the following possible cases.

(1) There exists $x \in S$ such that $d(x) > n$.

(1a) If $k = d(x) > n$ is finite, then $1/k < 1/n$, so we can find an element $t_0 \in \mathbb{T}$ such that $kt_0 = f(kx) \pmod 1$ and $|t_0 - s_n| < 1/n$. Define the mapping $h : H \cup \{x\} \to \mathbb{T}$ by $h|_H = f|_H$ and $h(x) = t_0$. We have $kh(x) = kt_0 = f(kx) = h(kx) \pmod 1$, so h admits an extension to a homomorphism of G to $\mathbb{T}$ (which we denote by the same letter h). Clearly, $h \in W_1 \cap U_n \neq \emptyset$.

(1b) If $d(x) = \infty$, we simply put $h(y_i) = f(y_i)$ for $i = 1, \ldots, m$, and $h(x) = s_n$. Since $kx \notin H$ for each $k \in \mathbb{N}$, h extends to a homomorphism $G \to \mathbb{T}$, so $h \in W_1 \cap U_n$.

(2) $d(x) \leqslant n$ for all $x \in S$.

Since S is infinite, we can assume without loss of generality that all numbers $d(x)$ are equal, say, to $d \in \mathbb{N}$. Let us note that the set $kT = \{kx\colon x \in T\}$ is infinite for every infinite subset T of S and $k \in N$. Indeed, suppose the contrary and choose an infinite set $T_0 \subseteq T$ and an element $z_0 \in G$ such that $kx = z_0$ for all $x \in T_0$. Pick $x_0 \in T_0$. Then $k(x - x_0) = 0$ for each $x \in T_0$, and hence $T_0 \subseteq x_0 + G[k]$. Therefore, the set $S \cap (x_0 + G[k]) \supseteq T_0$ is infinite, a contradiction.

We claim that at least one of the elements $y_1, \ldots, y_m$ has infinite order. Suppose to the contrary that the order d_i of y_i is finite for each $i = 1, \ldots, m$. Put $D = d_1 \cdots d_m$. Then for every $x \in S$, we have $dx \in H$, whence $Ddx = 0_G$. This immediately implies that $S \subseteq G[Dd]$, a contradiction.

Suppose that the elements $y_1, \ldots, y_p$ have infinite order and $y_{p+i}, \ldots, y_m$ have finite orders $d_{p+i}, \ldots, d_m$, respectively, where $1 \leqslant p \leqslant m$. Let $S = \{x_j\colon j \in \omega\}$. For every $j \in \omega$, there exist integers $k_{j,1}, \ldots, k_{j,m}$ such that $dx_j = \sum_{i=1}^{m} k_{j,i} y_i$ and $0 \leqslant k_{j,i} < d_i$ for each $i = p + 1, \ldots, m$. Again, we can find an infinite subset J of ω and non-negative integers $k_{p+1}, \ldots, k_m$ such that $k_{j,i} = k_i$ whenever $j \in J$ and $p + 1 \leqslant i \leqslant m$. For every $j \in J$, put

$r(j) = \sum_{i=1}^{p} |k_{j,i}|$. Since the set $\{dx_j\colon\ j \in J\}$ is infinite, $r(j) > 1/\delta$ for some $j \in J$. For this j, one can choose $t_1, \ldots, t_p \in \mathbb{T}$ satisfying

(i) $|t_i - f(y_i)| < \delta$ for $i = 1, \ldots, p$;

(ii) $\sum_{i=1}^{p} k_{j,i} t_i = ds_n - \sum_{i=p+1}^{m} k_i f(y_i) \pmod 1$.

It remains to put $h(y_i) = t_i$ for $i = 1, \ldots, p, h(y_i) = f(y_i)$ for $i = p+1, \ldots, m$ (this defines h on H) and $h(x_j) = s_n$. Then by (ii), we have

$$h(dx_j) = \sum_{i=1}^{p} k_{j,i} t_i + \sum_{i=p+1}^{m} k_i f(y_i) = ds_n = dh(x_j) \pmod 1,$$

and hence h admits an extension to a homomorphism of G to $\mathbb{T}$. Since $x_j \in S$, from (i) and our definition of h it follows that $h \in W_1 \cap U_n \neq \emptyset$. This completes the proof. □

It is clear that every infinite subset S of an Abelian almost torsion-free group G satisfies the condition $|S \cap (x + G[n])| \leqslant |G[n]| < \omega$ for all $x \in G$ and $n \in \mathbb{N}$. Therefore, the next result follows from Lemma 3.3 and the Baire category theorem.

Lemma 3.4. *Let $\{S_\alpha\colon\ \alpha < \kappa\}$ be a family of infinite subsets of an Abelian almost torsion-free group G, and let*

$$H = \bigl\{f \in G^*\colon\ f(S_\alpha) \text{ is dense in } \mathbb{T} \text{ for each } \alpha < \kappa\bigr\}.$$

(a) *If $\kappa < \omega_1$, then H is dense in G^*.*

(b) *If $\kappa < \mathfrak{c}$, then the conclusion in* (a) *remains valid under* MA.

Proof. For every $\alpha < \kappa$, put $H_\alpha = \{h \in G^*\colon\ h(S_\alpha) \text{ is dense in } \mathbb{T}\}$. Then $H = \bigcap_{\alpha<\kappa} H_\alpha$.

(a) Every H_α is an intersection of countably many open dense subsets of G^* by Lemma 3.3. If $\kappa < \omega_1$, H is also such an intersection. Since the group G^* is compact, the Baire category theorem implies that H is dense in G^*.

(b) MA is equivalent to the statement that a compact Hausdorff space of countable cellularity cannot be a union of less than $\mathfrak{c}$ nowhere dense sets [25]. Since the compact group G^* is dyadic [26,43], it has countable cellularity. By Lemma 3.3, the complement $G^* \setminus H_\alpha$ is the union of countably many nowhere dense sets for each $\alpha < \kappa$, so $G^* \setminus H$ is the union of at most κ nowhere dense sets. Thus MA implies that H is dense in G^*. □

We shall also use the following simple result concerning Abelian groups:

Lemma 3.5. *Let N be a subgroup of an Abelian almost torsion-free group G. Then there exists a subgroup K of G such that $N \subseteq K$, $|K| \leqslant |N| \cdot \omega$ and G/K is torsion-free.*

Proof. Since G is almost torsion-free, the subgroup $G[n] = \{x \in G\colon\ nx = 0\}$ is finite for each $n \in \mathbb{N}$. Therefore, the torsion subgroup $H = \bigcup_{n=1}^{\infty} G[n]$ of G is countable. For every $g \in G$, put

$$H(g) = \bigl\{x \in G\colon\ \exists n \in \mathbb{Z} \setminus \{0\} \text{ such that } nx = g\bigr\}.$$

It is clear that $|H(g)| \leqslant |H| \cdot \omega \leqslant \omega$. Let us define $K = \bigcup\{H(g)\colon g \in N\}$. Then $N \subseteq K$, K is symmetric and $|K| \leqslant |N| \cdot \omega$. In addition, if $x, y \in K$, then $mx \in N$ and $ny \in N$ for some $m, n \in \mathbb{Z} \setminus \{0\}$, whence $mn(x + y) = mnx + mny \in N$. This proves that $x + y \in K$, i.e., K is a subgroup of G. Finally, if $x \in G$ and $mx = y \in K$ for some integer $m \neq 0$, then $y \in H(g)$ for some $g \in N$, and hence there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $ny = g$. But then $nmx = ny = g$, whence $x \in H(g) \subseteq K$. This proves that the quotient group G/K is torsion-free. $\square$

Proof of Theorem 3.1. In a sense, the construction that follows is close to the construction of a countably compact Hausdorff group topology on the free Abelian group with $\mathfrak{c}$ generators given in [38], but our task here is more complicated. We will construct a monomorphism $h : G \to \mathbb{T}^{\mathfrak{c}}$ satisfying (1) and (2) of Lemma 3.2, thus obtaining (a) of the theorem. For every $\nu < \mathfrak{c}$, let $p_\nu : \mathbb{T}^{\mathfrak{c}} \to \mathbb{T}_\nu$ be the projection to the νth factor. Given a subset A of $\mathfrak{c}$, we shall also use π_A to denote the projection of $\mathbb{T}^{\mathfrak{c}}$ onto $\mathbb{T}^A$. In particular, π_α is the projection of $\mathbb{T}^{\mathfrak{c}}$ to $\mathbb{T}^\alpha$. To guarantee (1) of Lemma 3.2, we shall take care of the following two strong properties of $h(G)$:

(A) $h(G)$ is an HFD subgroup of $\mathbb{T}^{\mathfrak{c}}$, i.e., for every countably infinite subset S of G there exists $\alpha < \mathfrak{c}$ such that $\pi_{\mathfrak{c}\setminus\alpha}(h(S))$ is dense in $T^{\mathfrak{c}\setminus\alpha}$;

(B) $\pi_\alpha(h(G)) = \mathbb{T}^\alpha$ for each $\alpha < \mathfrak{c}$.

From (A) it immediately follows that $h(G)$ does not contain non-trivial convergent sequences, and by the argument [20, p. 204], the combination of (A) and (B) implies that $h(G)$ is countably compact. In other words, (b) of the theorem follows from (A) and (B). In addition, $h(G)$ is dense in $\mathbb{T}^{\mathfrak{c}}$ (this follows from (B)). It remains to note that every dense countably compact subset of $\mathbb{T}^{\mathfrak{c}}$ is connected and locally connected. Indeed, the projections of $h(G)$ cover all countable faces of $\mathbb{T}^{\mathfrak{c}}$, so $h(G)$ is connected by a lemma of [37]. A similar argument gives local connectedness of $h(G)$. Indeed, let $P = \prod_{\alpha<\mathfrak{c}} P_\alpha$ be a product of closed intervals $P_\alpha \subseteq \mathbb{T}$, where only finitely many of them are different from $\mathbb{T}$. Again, the projections of the set $P \cap h(G)$ cover all countable faces of the product space P, so $P \cap h(G)$ is connected. Concluding, (c) of the theorem also follows from (B).

Our construction requires some preliminary work. By Lemma 3.5, the group G can be represented as the strictly increasing union $G = \bigcup_{\omega\leqslant\nu<\mathfrak{c}} N_\nu$ of its subgroups N_ν, where N_ω is the torsion part of G, $|N_\nu| = |\nu|$, G/N_ν is torsion-free if $\omega < \nu < \mathfrak{c}$, and $N_\nu = \bigcup_{\omega\leqslant\mu<\nu} N_\mu$ if ν is a limit ordinal. For every $\alpha < \mathfrak{c}$, we shall define a homomorphism $h_\alpha : G \to \mathbb{T}$ that in turn will enable us to take $h = \Delta_{\alpha<\mathfrak{c}} h_\alpha$, the diagonal product of the homomorphisms h_α. In fact, at the step $\alpha < \mathfrak{c}$ of the construction we shall define the family $\{h_{\beta,\nu}\colon \beta < \alpha,\ \omega \leqslant \nu \leqslant \alpha\}$, where $h_{\beta,\nu}$ is a homomorphism of N_ν to $\mathbb{T}$. In addition, the homomorphism $h_{\beta,\nu}$ will extend $h_{\beta,\mu}$ whenever $\omega \leqslant \mu < \nu$, so the restriction of h_β to N_ν will coincide with $h_{\beta,\nu}$ for all $\beta, \nu < \mathfrak{c}$, $\nu \geqslant \omega$. Since the subgroup N_ω of G is countable, we can find homomorphisms $h_{k,\omega} : N_\omega \to \mathbb{T}$, $k \in \omega$ which separate the elements of N_ω (in other words, for every $x \in N_\omega \setminus \{0\}$ there must exist $k \in \omega$ such that $h_{k,\omega}(x) \neq 0$).

Since $|\tau| = \mathfrak{c}$, there exists an enumeration $\{F_\alpha\colon \alpha < \mathfrak{c}\}$ of all closed subsets of (G, τ) having the size $\mathfrak{c}$. For every $x \in G$, denote by $\xi(x)$ the minimal ordinal $\xi \in \mathfrak{c} \setminus \omega$ such that $x \in N_\xi$. Note that $\xi(x)$ is either non-limit or $\xi(x) = \omega$. This enables us to define the sets

$$A_\alpha = \bigl\{\xi(x)\colon x \in F_\alpha\bigr\},$$

$\alpha < \mathfrak{c}$. Clearly, $|A_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. Consider the family $\{A_\alpha\colon \alpha < \mathfrak{c}\}$. It is easy to see that there exists a family $\{B_\alpha\colon \alpha < \mathfrak{c}\}$ satisfying the following conditions for all α, β with $\beta < \alpha < \mathfrak{c}$:

(i) $B_\alpha \subseteq A_\alpha$;
(ii) $|B_\alpha| = \mathfrak{c}$;
(iii) $B_\alpha \cap B_\beta = \emptyset$;
(iv) $L = \bigcup_{\alpha<\mathfrak{c}} B_\alpha = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$.

Note that $\omega \in L$ and $L \setminus \{\omega\}$ coincides with the set of all infinite non-limit ordinals less than $\mathfrak{c}$. We claim that there exists a surjective function $\varphi : L \to \mathfrak{c} \times \mathfrak{c}$ satisfying

(v) if $\varphi(\alpha) = (\beta, \delta)$, then $\alpha \in B_\beta$ and $\delta \leqslant \alpha$;
(vi) $\varphi(\omega) = (\beta, 0)$ for some $\beta < \mathfrak{c}$.

Indeed, for every $\beta < \mathfrak{c}$, let $\psi_\beta : \mathfrak{c} \to B_\beta$ be the order preserving isomorphism. It is clear that $\gamma \leqslant \psi_\beta(\gamma)$ for each $\gamma < \mathfrak{c}$. Put $\varphi(\alpha) = (\beta, \psi_\beta^{-1}(\alpha))$ for each $\alpha \in B_\beta$. One easily verifies that the function φ satisfies (v) and $\varphi(L) = \mathfrak{c} \times \mathfrak{c}$. In addition, there exists $\beta < \mathfrak{c}$ such that $\omega \in B_\beta$. Then $\omega = \min B_\beta$, whence it follows that $\psi_\beta(0) = \omega$. Therefore, $\varphi(\omega) = (\beta, 0)$. This implies (vi).

Denote by $\mathcal{B}$ the countable base for $\mathbb{T}$ consisting of open intervals with rational end-points. Let $\mathcal{U}$ be the family of all canonical open sets O in $\mathbb{T}^\mathfrak{c}$ such that $p_\alpha(O) \in \mathcal{B}$ for each $\alpha < \mathfrak{c}$. There exists an enumeration $\mathcal{U} = \{O_\delta\colon \delta < \mathfrak{c}\}$ such that $\operatorname{coord}(O_\delta) \subseteq \delta$ for each $\delta < \mathfrak{c}$, where $\operatorname{coord}(O_\delta) = \{\alpha < \mathfrak{c}\colon p_\alpha(O_\delta) \neq \mathbb{T}\}$. Clearly, we have to put $O_0 = \mathbb{T}^\mathfrak{c}$.

We now need several additional enumerations. Let $\{S_\mu\colon \omega \leqslant \mu < \mathfrak{c}\}$ be an enumeration of all countably infinite subsets of G such that $\operatorname{supp}(S_\mu) = \{\xi(x)\colon x \in S_\mu\} \subseteq \mu$ for each $\mu \geqslant \omega$ (this is to deal with (A)). Denote by Σ the subgroup of $\mathbb{T}^\mathfrak{c}$ consisting of all points $t \in \mathbb{T}^\mathfrak{c}$ satisfying $|\{\alpha < \mathfrak{c}\colon t(\alpha) \neq 0\}| < \mathfrak{c}$. Since MA implies that $2^\kappa = \mathfrak{c}$ for every infinite cardinal $\kappa < \mathfrak{c}$, we can enumerate $\Sigma = \{b_\alpha\colon \alpha < \mathfrak{c}\}$ in such a way that every element $b \in \Sigma$ appears in this enumeration $\mathfrak{c}$ times (this is to deal with (B)).

We will define a family $\{h_{\alpha,\nu}\colon \alpha < \mathfrak{c},\ \omega \leqslant \nu < \mathfrak{c}\}$ satisfying the following conditions for all $\alpha, \nu < \mathfrak{c}$, $\nu \geqslant \omega$:

(1) $h_{\alpha,\nu} : N_\nu \to \mathbb{T}$ is a homomorphism and $h_{\alpha,\nu}$ extends $h_{\alpha,\mu}$ if $\omega \leqslant \mu < \nu$;
(2) the image $(\Delta_{\mu\leqslant\gamma<\alpha} h_{\gamma,\nu})(S_\mu)$ is dense in $\mathbb{T}^{\alpha\setminus\mu}$ whenever $\mu < \alpha$ and $\omega \leqslant \mu < \nu$;
(3) there exists a point $x \in N_{\nu+1} \setminus N_\nu$ such that $h_{\gamma,\nu+1}(x) = b_\nu(\gamma)$ for each $\gamma < \nu$;
(4) if $\alpha \in L$, $\varphi(\alpha) = (\beta, \delta)$ and $\operatorname{coord}(O_\delta) = \{\mu_1, \ldots, \mu_k\}$, then there exists a point $y \in F_\beta$ such that $\xi(y) = \alpha$ and $h_{\mu_i,\alpha}(y) \in p_{\mu_i}(O_\delta)$ for each $i = 1, \ldots, k$.

Suppose that the family $\{h_{\alpha,\nu}\colon \alpha < \mathfrak{c},\ \omega \leqslant \nu < \mathfrak{c}\}$ satisfying (1)–(4) has been constructed. By (1), for every $\alpha < \mathfrak{c}$ there exists a homomorphism $h_\alpha : G \to \mathbb{T}$ whose restriction to N_ν coincides with $h_{\alpha,\nu}$ for each $\nu \in \mathfrak{c} \setminus \omega$. Denote by h the diagonal product of the homomorphisms h_α, $\alpha < \mathfrak{c}$. We claim that the homomorphism $h : G \to \mathbb{T}^\mathfrak{c}$ satisfies (1) and (2) of Lemma 3.2.

First, from the above conditions (1) and (2) it follows that h is a monomorphism. Indeed, let $x \in G \setminus \{0\}$ be arbitrary. If $nx = 0$ for some $n \in N$, then $x \in N_\omega$, and hence $h_{k,\omega}(x) \neq 0$ for some $k \in \omega$. If $x \in G \setminus N_\omega$, then $\{nx\colon n \in \mathbb{Z}\} = S_\mu$ for some $\mu < \mathfrak{c}$, so $h_{\mu,\mu+1}(S_\mu)$ is dense in $\mathbb{T}$ by virtue of (2). This and (1) together imply that $h_\mu(x) = h_{\mu,\mu+1}(x) \neq 0$. In any event, $x \notin \ker h$.

It is easy to see that $h(G)$ satisfies (A). Indeed, if S is a countably infinite subset of G, there exists $\mu < \mathfrak{c}$ such that $S = S_\mu$. For every α satisfying $\mu < \alpha < \mathfrak{c}$, let $f_{\mu,\alpha} = \Delta_{\mu \leqslant \gamma < \alpha} h_{\gamma,\alpha}$ be the diagonal product of the family $\{h_{\gamma,\alpha}\colon \mu \leqslant \gamma < \alpha\}$. From the equality $\pi_{\alpha\setminus\mu} \circ h = f_{\mu,\alpha}$ and (2) it follows that $\pi_{\alpha\setminus\mu}(h(S_\mu))$ is dense in $\mathbb{T}^{\alpha\setminus\mu}$ for each α with $\mu < \alpha < \mathfrak{c}$, and hence $\pi_{\mathfrak{c}\setminus\mu}(h(S_\mu))$ is dense in $\mathbb{T}^{\mathfrak{c}\setminus\mu}$. In other words, $h(G)$ is an *HFD* subgroup of $\mathbb{T}^{\mathfrak{c}}$.

Let us show that $h(G)$ satisfies (B). Suppose that $\alpha < \mathfrak{c}$ and $z \in \mathbb{T}^\alpha$. There exists $\nu < \mathfrak{c}$ such that $\alpha \leqslant \nu$ and $z(\gamma) = b_\nu(\gamma)$ for each $\gamma < \alpha$. By (3), one can find a point $x \in N_{\nu+1} \setminus N_\nu$ such that $h_{\gamma,\nu+1}(x) = b_\nu(\gamma)$ for each $\gamma < \nu$. This and the definition of h together imply that $\pi_\alpha(h(x)) = z$. Since z is an arbitrary point of $\mathbb{T}^\alpha$, we have proved that $\pi_\alpha(h(G)) = \mathbb{T}^\alpha$.

As the combination of (A) and (B) implies countable compactness of $h(G)$, we conclude that $h(G)$ satisfies (1) of Lemma 3.2. It remains to show that $h(G)$ also satisfies (2) of Lemma 3.2.

To this end, it suffices to verify that for every $\beta < \mathfrak{c}$, the image $h(F_\beta)$ intersects every canonical open set O_δ in $\mathbb{T}^{\mathfrak{c}}$. There exists $\alpha \in L$ such that $\varphi(\alpha) = (\beta, \delta)$. Let $\operatorname{coord}(O_\delta) = \{\mu_1, \ldots, \mu_k\}$. Note that $\operatorname{coord}(O_\delta) \subseteq \delta \leqslant \alpha$ by (v), so $\mu_i < \alpha$ for each $i = 1, \ldots, k$. Apply (4) to find a point $y \in F_\beta$ such that $\xi(y) = \alpha$ and $h_{\mu_i,\alpha}(y) \in p_{\mu_i}(O_\delta)$ for each $i = 1, \ldots, k$. Since the restriction of h_{μ_i} to N_α coincides with $h_{\mu_i,\alpha}$, $1 \leqslant i \leqslant k$, the definition of h implies that $h(y) \in O_\delta$, i.e., $h(F_\beta) \cap O_\delta \neq \emptyset$.

Therefore, by Lemma 3.2, the topologies $\mathcal{T} = \{h^{-1}(U \cap h(G))\colon U \text{ is open in } \mathbb{T}^{\mathfrak{c}}\}$ and τ on G are independent.

Construction of the homomorphism h. Finally, we prove the existence of a family $\{h_{\alpha,\nu}\colon \alpha \in \mathfrak{c},\ \nu \in \mathfrak{c} \setminus \omega\}$ satisfying (1)–(4). There exists a family $\mathcal{H}_\omega = \{h_{k,\omega}\colon k \in \omega\}$ of homomorphisms of N_ω to $\mathbb{T}$ that separates the elements of N_ω. Since $\varphi(\omega) = (\beta, 0)$ for some $\beta < \mathfrak{c}$ (see (vi)) and $O_0 = \mathbb{T}^{\mathfrak{c}}$, we have $\operatorname{coord}(O_0) = \emptyset$, so (4) trivially holds for the family $\mathcal{H}_\omega$. The properties (1)–(3) are obvious.

Let α be an ordinal with $\omega < \alpha < \mathfrak{c}$, and suppose that for every $\beta < \alpha$, we have defined a family $\mathcal{H}_\beta = \{h_{\gamma,\nu}\colon \gamma < \beta,\ \omega \leqslant \nu \leqslant \beta\}$ satisfying (1)–(4). If α is limit, then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, and from (1) it follows that for every $\gamma < \alpha$ there exists a homomorphism $h_{\gamma,\alpha}\colon N_\alpha \to \mathbb{T}$ such that $h_{\gamma,\alpha}|_{N_\beta} = h_{\gamma,\beta}$ for each β with $\omega \leqslant \beta < \alpha$. Therefore, the family $\mathcal{H}_\alpha = \{h_{\gamma,\nu}\colon \gamma < \alpha,\ \omega \leqslant \nu \leqslant \alpha\}$ satisfies (1)–(4). It suffices, therefore, to consider the case when α is non-limit, say, $\alpha = \beta + 1$.

To define the family $\mathcal{H}_\alpha$, we have to extend the homomorphisms $h_{\gamma,\beta} \in \mathcal{H}_\beta$ (with $\gamma < \beta$) over N_α, thus obtaining the homomorphisms $h_{\gamma,\alpha}$. In addition, we have to construct a homomorphism $h_{\beta,\alpha}$ with the same domain N_α.

We start with extensions of the homomorphisms $h_{\gamma,\beta}$. Let $\varphi(\alpha) = (\gamma, \delta)$. Since $\alpha \in L$, there exists a point $y \in F_\gamma$ such that $\xi(y) = \alpha$ or, equivalently, $y \in N_\alpha \setminus N_\beta$. Let also $\operatorname{coord}(O_\delta) = \{\mu_1, \ldots, \mu_k\}$. From (v) and the choice of the enumeration of $\mathcal{U}$ it follows that $\operatorname{coord}(O_\delta) \subseteq \delta \leqslant \alpha$, whence $\mu_i < \alpha$ for each $i = 1, \ldots, k$. For every $i \leqslant k$, denote by $O_{\delta,i}$ the projection of O_δ to the factor $\mathbb{T}_{\mu_i}$. According to (4), we have to define for every $\gamma < \beta$ an extension $h_{\gamma,\alpha}$ of $h_{\gamma,\beta}$ satisfying

$$h_{\mu_i,\alpha}(y) \in O_{\delta,i} \quad \text{for each } i = 1, \ldots, k. \tag{$*$}$$

Also, we have to choose an element $x \in N_\alpha \setminus N_\beta$ satisfying (3), i.e.,

$$h_{\gamma,\alpha}(x) = b_\beta(\gamma) \quad \text{for each } \gamma < \beta. \tag{$**$}$$

Since G/N_β is torsion-free, there is no problem to take any $x \in N_\alpha \setminus N_\beta$ and define an extension $h_{\gamma,\alpha}$ of $h_{\gamma,\beta}$ satisfying $(**)$ for each $\gamma \in \beta \setminus \{\mu_1, \ldots, \mu_k\}$. Let us show that this is also possible for every $\gamma \in \{\mu_1, \ldots, \mu_k\}$, i.e., $(*)$ and $(**)$ are compatible. To this end, note that the system

$$\begin{cases} n \cdot z = a \\ z \in U \end{cases}$$

with $a \in \mathbb{T}$, $n \in \mathbb{N}$ and a non-empty set $U \in \mathcal{B}$ has a solution $z \in \mathbb{T}$ if $1/n$ is less than the length of U (we consider $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as the Abelian group with the sum operation modulo 1, so the total length of $\mathbb{T}$ is equal to 1). Choose $n \in \mathbb{N}$ such that $1/n$ is less than the length of $O_{\delta,i}$ for each $i \leqslant k$ and put $x = n \cdot y$. According to $(*)$ and $(**)$, we must have for every $i = 1, \ldots, k$:

$$\begin{cases} h_{\mu_i,\alpha}(ny) = nh_{\mu_i,\alpha}(y) = b_\beta(\mu_i) \\ h_{\mu_i,\alpha}(y) \in O_{\delta,i}. \end{cases}$$

By the above observation, for every $i = 1, \ldots, k$, it possible to choose a solution $z_i = h_{\mu_i,\alpha}(y)$ of this system. Then define a homomorphism $h_{\mu_i,\alpha} : N_\alpha \to \mathbb{T}$ satisfying $h_{\mu_i,\alpha}|_{N_\beta} = h_{\mu_i,\beta}$ and $h_{\mu_i,\alpha}(y) = z_i$, $i = 1, \ldots, k$.

Let us construct the homomorphism $h_{\beta,\alpha}$. For every $\mu < \beta$, denote by $f_{\mu,\beta}$ the diagonal product of the family $\{h_{\gamma,\beta}\colon \mu \leqslant \gamma < \beta\}$. Clearly, $f_{\mu,\beta}$ is a homomorphism of N_β to $\mathbb{T}^{\beta\setminus\mu}$. Let $\mathcal{U}_{\mu,\beta}$ be the family of all canonical open sets in $\mathbb{T}^{\beta\setminus\mu}$ whose projections to the factors belong to the standard base $\mathcal{B}$ of $\mathbb{T}$. Since $|\mathcal{U}_{\mu,\beta}| \leqslant |\beta|$ for each $\mu < \beta$, the cardinality of the family

$$\mathcal{K}_\beta = \left\{S_\mu \cap f_{\mu,\beta}^{-1}\bigl(U \cap f_{\mu,\beta}(S_\mu)\bigr)\colon U \in \mathcal{U}_{\mu,\beta},\ \omega \leqslant \mu < \beta\right\} \cup \{S_\beta\}$$

does not exceed $|\beta|$. Note that by (2) (with β instead of α), all elements of $\mathcal{K}_\beta$ are infinite. Therefore, we can apply Lemma 3.4(b) to find a homomorphism $h_{\beta,\alpha} : N_\alpha \to \mathbb{T}$ such that $h_{\beta,\alpha}(K)$ is dense in $\mathbb{T}$ for each $K \in \mathcal{K}_\beta$. This definition implies the validity of (2) for $\alpha = \beta + 1$. Indeed, the density of $h_{\beta,\alpha}(S_\beta)$ in $\mathbb{T}^{\alpha\setminus\beta} = \mathbb{T}$ is clear. For every $\mu < \beta$, let $f_{\mu,\alpha} = \Delta_{\mu\leqslant\gamma<\alpha} h_{\gamma,\alpha}$ be the diagonal product of the homomorphisms $h_{\gamma\alpha}$, $\mu \leqslant \gamma < \alpha$. We have to show that $f_{\mu,\alpha}(S_\mu)$ is dense in $\mathbb{T}^{\alpha\setminus\mu}$. Consider a nonempty canonical open set $U \in \mathcal{U}_{\mu,\alpha}$ in $\mathbb{T}^{\alpha\setminus\mu}$. Then $U = U_1 \times V$, where $U_1 \in \mathcal{U}_{\mu,\beta}$ is a canonical open set in $\mathbb{T}^{\beta\setminus\mu}$ and V is open in $\mathbb{T}_\beta$. By (2), $f_{\mu,\beta}(S_\mu)$ is dense in $\mathbb{T}^{\beta\setminus\mu}$. Therefore, $S =$

$S_\mu \cap f_{\mu,\beta}^{-1}(U_1 \cap f_{\mu,\beta}(S_\mu))$ belongs to the family $\mathcal{K}_\beta$, and hence $h_{\beta,\alpha}(S)$ is dense in $\mathbb{T}$. Choose a point $x \in S$ such that $h_{\beta,\alpha}(x) \in V$. Then $f_{\mu,\alpha}(x) = (f_{\mu,\beta}(x), h_{\beta,\alpha}(x)) \in U_1 \times V$, whence $f_{\mu,\alpha}(S_\mu) \cap U \neq \emptyset$. This finishes our recursive construction. Clearly, the family $\mathcal{H}_\alpha = \{h_{\gamma,\nu}\colon \gamma < \alpha,\ \omega \leqslant \nu \leqslant \alpha\}$ satisfies (1)–(4). The theorem is proved. □

It is well known that the number of open sets in a space X does not exceed $2^{w(X)}$. Since MA implies that $2^\lambda = \mathfrak{c}$ for each infinite cardinal $\lambda < \mathfrak{c}$ (see [32]), Theorem 3.1 has the following corollary.

Corollary 3.6. *Under* MA, *every Abelian almost torsion-free topological group G satisfying $w(G) < \mathfrak{c} = |G|$ admits an independent group topology.*

Combining Theorem 3.1 and Proposition 2.4, we obtain the result concerning basic topological groups:

Corollary 3.7. *Under* MA, *each of the groups* $\mathbb{R}$, $\mathbb{T}$, $\mathbb{C}$ *and* $\mathbb{C}^*$ (*with its usual Euclidean topology*) *admits an independent group topology. Such a topology is necessarily countably compact and makes all non-trivial sequences divergent.*

Remark 3.8. (a) The use of topological properties of the group (G, τ) in the proof of Theorem 3.1 was very modest: we only used the fact that the family of all closed subsets of (G, τ) has cardinality $\mathfrak{c}$. Therefore, Theorem 3.1 remains valid if the group (G, τ) of cardinality $\mathfrak{c}$ is hereditarily separable. In particular, this is the case when (G, τ) has a countable network.

(b) Theorem 3.1 cannot be extended to all Abelian torsion groups. In fact, even the Abelian groups of finite order present problems. Indeed, let p, q be distinct primes. Denote by $H(p)$ and $H(q)$ two infinite Abelian groups of order p and q, respectively. Then no Hausdorff group topologies τ and $\mathcal{T}$ on the group $G = H(p) \oplus H(q)$ are independent. To see this, note that the subgroup $H(p) \times \{0\}$ of G is *algebraically closed* in G being the kernel of the homomorphism $f : G \to G$ defined by $f(x) = px$. Since the homomorphism f is continuous in every group topology on G, the set $H(p) \times \{0\}$ must be closed in both groups $(G, \mathcal{T})$ and (G, τ). However, we shall see in Section 4 that Theorem 3.1 remains valid for Abelian groups of a prime order.

(c) If one makes the use of CH instead of MA, the group topology $\mathcal{T}$ on G in Theorem 3.1 can additionally be chosen hereditarily separable and hereditarily normal. These two properties of $\mathcal{T}$ also follow from (A) and (B) in the proof of Theorem 3.1 (for more details, see [19]). We do not know, however, whether Theorem 3.1 remains valid under $\mathrm{MA}_{\mathrm{countable}}$.

Clearly, Theorem 3.1 implies that all proper unconditionally closed subsets of an Abelian almost torsion-free group of size $\mathfrak{c}$ are finite. However, the proof of the theorem depends on MA. It turns out that our conclusion about unconditionally closed subsets of such groups remains valid in ZFC no matter how big the groups are (see Theorem 3.13). The key notion in the proof of this fact will be that of *potentially dense subset* of an abstract group introduced by Markov in [28].

Definition 3.9. A subset A of a group G is *potentially dense* in G if there exists a Hausdorff group topology τ on G such that A is dense in (G,τ).

It was shown in [28, p. 304] that every infinite subset of the group $\mathbb{Z}$ is potentially dense in $\mathbb{Z}$. The same conclusion also follows from [11, Lemma 5.2]: for every infinite subset A of $\mathbb{Z}$, there exists $t \in \mathbb{T}$ such that the set $A\cdot t = \{n\cdot t\colon n \in A\}$ is dense in $\mathbb{T}$. Therefore, if $h:\mathbb{Z}\to\mathbb{T}$ is the monomorphism defined by $h(n) = n\cdot t$ for each $n\in\mathbb{Z}$, then the isomorphic copy $h(\mathbb{Z})$ of $\mathbb{Z}$ contains $h(A)$ as a dense subset. We will prove a considerably more general result for Abelian almost torsion-free groups:

Theorem 3.10. *For every infinite subset S of an Abelian almost torsion-free group G with $|G|\leqslant\mathfrak{c}$, there exists a second countable precompact Hausdorff group topology τ on G such that S is dense in (G,τ). In particular, S is potentially dense in G.*

Again, we need a couple of auxiliary results the first of which is of algebraic character.

Lemma 3.11. *For every countable subgroup N of $\mathbb{T}$, there exists a subgroup P of $\mathbb{T}$ isomorphic to the direct sum $\bigoplus_{\xi<\mathfrak{c}}\mathbb{Q}_\xi$ of $\mathfrak{c}$ copies of the group $\mathbb{Q}$ such that $P\cap N=\{0\}$.*

Proof. Let $\pi:\mathbb{R}\to\mathbb{R}/\mathbb{Z}=\mathbb{T}$ be the canonical homomorphism. Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ and choose a Hamel basis $\{r_\alpha\colon \alpha<\mathfrak{c}\}$ of $\mathbb{R}$ over $\mathbb{Q}$ such that $r_0=1$. Clearly, $\mathbb{R}\cong\bigoplus_{\alpha<\mathfrak{c}} r_\alpha\cdot\mathbb{Q}$. Since $\ker\pi=\mathbb{Z}\subseteq\mathbb{Q}=r_0\cdot\mathbb{Q}$, we conclude that $\mathbb{T}=\mathbb{R}/\mathbb{Z}\cong\mathbb{Q}/\mathbb{Z}\oplus(\bigoplus_{0<\alpha<\mathfrak{c}}\mathbb{Q}_\alpha)$, where each $\mathbb{Q}_\alpha$ is isomorphic with $\mathbb{Q}$. It remains to note that every countable subgroup N of $\mathbb{T}$ is contained in $\mathbb{Q}/\mathbb{Z}\oplus(\bigoplus_{\alpha\in A}\mathbb{Q}_\alpha)$ for a countable set $A\subseteq\mathfrak{c}$, so the conclusion of the lemma is immediate. □

The next lemma is a special case of Theorem 3.10 restricted to countable groups.

Lemma 3.12. *For every infinite subset S of a countable Abelian almost torsion-free group K, there exists a monomorphism $f:K\to\mathbb{T}^\omega$ such that $f(S)$ is dense in $\mathbb{T}^\omega$.*

Proof. Let $K=\{x_n\colon n\in\omega\}$, where $x_0=0_K$. By Lemma 3.3, there exists a homomorphism $h_0:K\to\mathbb{T}$ such that $h_0(S)$ is dense in $\mathbb{T}$. Suppose that for some $n\in\omega$, we have defined homomorphisms h_k, $0\leqslant k\leqslant n$, satisfying

(1) $f_n(S)$ is dense in $\mathbb{T}^{n+1}$, where f_n is the diagonal product of $h_0,\dots,h_n$;

(2) $h_k(x_k)\neq 0$ if $0<k\leqslant n$.

Denote by $\mathcal{B}_n$ a countable base of $\mathbb{T}^{n+1}$ and put

$$\mathcal{L}_n=\bigl\{S\cap f_n^{-1}\bigl(U\cap f_n(S)\bigr)\colon U\in\mathcal{B}_n\bigr\}.$$

From (1) it follows that all elements of $\mathcal{L}_n$ are infinite. Lemma 3.4(a) implies that

$$H_n=\bigl\{h\in K^*\colon h(L) \text{ is dense in } \mathbb{T} \text{ for each } L\in\mathcal{L}_n\bigr\}$$

is dense in the Pontryagin dual K^* of K. The set $W = \{h \in K^*\colon h(x_{n+1}) \neq 0\}$ is open in K^* and non-empty, so there exists a homomorphism $h_{n+1} \in W \cap H_n$. An easy verification shows that the family $\{h_k\colon 0 \leqslant k \leqslant n+1\}$ satisfies (1) and (2) with $n+1$ instead of n. This finishes our construction.

It remains to define f as the diagonal product of the family $\{h_k\colon k \in \omega\}$. $\square$

Proof of Theorem 3.10. Let S be an infinite subset of an Abelian almost torsion-free group G satisfying $|G| \leqslant \mathfrak{c}$. We can assume that $|S| = \omega < |G| = \kappa$, otherwise the conclusion of the theorem follows from Lemma 3.12. Denote by $H(G)$ the divisible hull of G, i.e., the minimal divisible Abelian group that contains G as a subgroup [31]. It is easy to verify that $|H(G)| = |G|$ and $r_p(H(G)) = r_p(G)$ for each prime p, so that $H(G)$ is also almost torsion-free. It suffices to find a second countable precompact Hausdorff group topology τ on $H(G)$ that makes S dense in $(H(G), \tau)$, and hence the assumption that G is divisible itself does not reduce generality.

Being divisible, the group G is isomorphic to the direct sum $G \cong \bigoplus_{\xi<\kappa} N_\xi$ of κ summands each of which is either $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$ for a prime p. Since G is almost torsion-free, we can assume that $S \subseteq \bigoplus_{\xi<\omega} N_\xi = K$ and $N_\xi = \mathbb{Q}$ for each $\xi \geqslant \omega$. Apply Lemma 3.12 to find a monomorphism $f : K \to \mathbb{T}^\omega$ such that $f(S)$ is dense in $\mathbb{T}^\omega$. Let $\tilde{f} : G \to \mathbb{T}^\omega$ be the homomorphism extending f which is trivial on $L = \bigoplus_{\omega\leqslant\xi<\kappa} N_\xi$. Denote by N the projection of the group $f(K)$ onto the first factor in $\mathbb{T}^\omega$ and apply Lemma 3.11 to find a subgroup P of $\mathbb{T} = \mathbb{T}_0$ isomorphic with L such that $N \cap P = \{0\}$. Let $g_0 : L \to P$ be such an isomorphism. Denote by g the homomorphism of G to $\mathbb{T}^\omega$ which is trivial on K and complements g_0 by zeros on L, i.e., $g(x) = (g_0(x), 0, 0, \ldots)$ for each $x \in L$. Consider the homomorphism $\varphi : G \to \mathbb{T}^\omega$, $\varphi = f + g$. From $N \cap P = \{0\}$ it follows that φ is injective, and the density of $\varphi(S)$ in $\mathbb{T}^\omega$ is immediate because f, $\tilde{f}$ and φ coincide on K. It remains to define the second countable precompact Hausdorff group topology τ on G by

$$\tau = \bigl\{\varphi^{-1}\bigl(U \cap \varphi(G)\bigr)\colon U \text{ is open in } \mathbb{T}^\omega\bigr\}.$$

The set S is clearly dense in (G, τ). $\square$

It is well known that the group $\mathbb{Z}$ can be embedded to $\mathbb{T}^{\mathfrak{c}}$ as a dense subgroup [23]. This suggests the following conjecture the authors have been unable to prove or refute: every infinite subset of an Abelian almost torsion-free group G with $|G| \leqslant 2^{\mathfrak{c}}$ is potentially dense in G (see Problem 6.5).

Let us show that Abelian almost torsion-free groups do not contain non-trivial unconditionally closed sets.

Theorem 3.13. *Proper unconditionally closed subsets of an Abelian almost torsion-free group are finite.*

Proof. Let S be a proper infinite subset of an Abelian almost torsion-free group G. There exists a countable subgroup K of G such that $S' = S \cap K$ is a proper infinite subset of K.

By Lemma 3.12, there exists a monomorphism $f : K \to \mathbb{T}^\omega$ such that $f(S')$ is dense in $\mathbb{T}^\omega$. Then

$$\tau = \bigl\{ f^{-1}\bigl(U \cap f(K)\bigr) \colon U \text{ is open in } \mathbb{T}^\omega \bigr\}$$

is a Hausdorff group topology on K and S' is dense in (K, τ). Then the family

$$\mathcal{B} = \bigl\{ xV \colon x \in G,\ V \in \tau \bigr\}$$

forms a base of a Hausdorff group topology $\mathcal{T}$ on G that induces the topology τ on K. If S were closed in $(G, \mathcal{T})$, the intersection $S' = S \cap K$ would be closed in (K, τ), a contradiction. Therefore, S cannot be unconditionally closed in G. □

The problem of existence of countably compact group topologies is very far from being solved even for Abelian groups. No examples of "non-trivial" countably compact groups of cardinality greater than $\mathfrak{c}$ are known (except for "trivial" Σ-products of compact groups or direct products of ω-bounded groups). It is still an open problem whether there exists a countably compact group topology on a free Abelian group of cardinality $> \mathfrak{c}$ (see [10, Question 5.16] or [5, Question 3.9.4]). Even the case of Abelian groups of size $\mathfrak{c}$ presents difficulties. The existence of a countably compact group topology on the free Abelian group with $\mathfrak{c}$ generators was established in [38] under CH. Later on, Tomita [41, 42] presented several examples of countably compact topological groups with different additional properties under MA. We obtain as a by-product the following general result for Abelian almost torsion-free groups.

Corollary 3.14. *Under* MA, *every Abelian almost torsion-free group of size* $\mathfrak{c}$ *admits a countably compact group topology without non-trivial convergent sequences.*

Proof. By Theorem 3.10, every Abelian almost torsion-free group G admits a second countable group topology τ. Apply Theorem 3.1 to conclude that there exists a group topology $\mathcal{T}$ on G independent of τ which is countably compact and does not contain non-trivial convergent sequences by Proposition 2.4. □

4. Groups of a prime order

The results of the previous section have their counterparts for Abelian groups of a prime order, but several changes are required in this case. First, if G is an Abelian group of a prime order p, then $h(G) \subseteq Z(p)$ for each character $h \in G^*$, where $\mathbb{Z}(p)$ is the cyclic subgroup of $\mathbb{T}$ generated by the element $1/p$. Since $\mathbb{Z}(p)$ is finite, we have to replace $\mathbb{T}$ by $\mathbb{Z}(p)^\omega$ for such a group G. In particular, the group of homomorphisms of G to $\mathbb{Z}(p)^\omega$ is naturally identified with $(G^*)^\omega$ (see Lemmas 4.1 and 4.2).

In the proof of the following lemma (which is analogous to Lemma 3.3 established for Abelian almost torsion-free groups) we shall use a simple algebraic fact: any subgroup N of an Abelian group G of a prime order p is a direct summand of G [31]. In particular, given a homomorphism $f : N \to \mathbb{Z}(p)$ and elements $x \in G \setminus N$ and $g \in \mathbb{Z}(p)$, there exists a homomorphism $h : G \to \mathbb{Z}(p)$ such that $h|_N = f$ and $h(x) = g$.

Lemma 4.1. *Let S be an infinite subset of a discrete Abelian group G of prime order p. Then the set*

$$H_S = \left\{ (f_i)_{i\in\omega} \in (G^*)^\omega \colon\ (\Delta_{i\in\omega} f_i)(S) \textit{ is dense in } \mathbb{Z}(p)^\omega \right\}$$

is the intersection of countably many dense open sets in $(G^*)^\omega$.

Proof. For every $n \in \mathbb{N}$ and $g \in \mathbb{Z}(p)^{n+1}$, consider the set

$$U_g = \left\{ (f_i)_{i\in\omega} \in (G^*)^\omega \colon\ \exists x \in S \text{ such that } f_i(x) = g(i),\ i = 0, \ldots, n \right\}.$$

Clearly, the sets U_g are open in $(G^*)^\omega$. The equality

$$H_S = \bigcap \left\{ U_g \colon\ g \in \mathbb{Z}(p)^{n+1} \text{ for some } n \in \mathbb{N} \right\}$$

is also evident. It remains to show that U_g is dense in $(G^*)^\omega$ for each $g \in \mathbb{Z}(p)^{n+1}$, $n \in \mathbb{N}$. This is equivalent to saying that the set

$$V_g = \Big\{ (f_0, \ldots, f_n) \in (G^*)^{n+1} \colon\ \exists x \in S \text{ such that } \\ f_i(x) = g(i),\ i = 0, \ldots, n \Big\}$$

is dense in $(G^*)^{n+1}$. Let $n \in \mathbb{N}$ and $g \in \mathbb{Z}(p)^{n+1}$ be arbitrary. Consider a basic open set $W = W_0 \times \cdots \times W_n$ in $(G^*)^{n+1}$, where

$$W_i = W(\varphi_i, x_{i,1}, \ldots, x_{i,n_i}) = \left\{ f \in G^* \colon\ f(x_{i,k}) = \varphi_i(x_{i,k}),\ k = 1, \ldots, n_i \right\}$$

and $\varphi_i \in G^*$, $x_{i,k} \in G$ $(0 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant n_i)$. Denote by N the subgroup of G generated by the elements $x_{i,k}$ with $i \leqslant n$ and $1 \leqslant k \leqslant n_i$. Since G is an Abelian torsion group, the subgroup N is finite. Therefore, we can pick an element $x \in S \setminus N$. For every $i \leqslant n$, define a homomorphism $h_i : G \to \mathbb{Z}(p)$ satisfying $h_i|_N = \varphi_i|_N$ and $h_i(x) = g(i)$. Clearly, $(h_0, \ldots, h_n) \in W \cap V_g$, thus finishing the proof. □

The combination of Lemma 4.1 and the Baire category theorem implies an analog of Lemma 3.4:

Lemma 4.2. *Let $\{S_\alpha \colon \alpha < \kappa\}$ be a family of infinite subsets of an Abelian group G of prime order p. Then*:

(a) *if $\kappa < \omega_1$, the set*

$$H = \left\{ f \in (G^*)^\omega \colon\ f(S_\alpha) \textit{ is dense in } \mathbb{Z}(p)^\omega \textit{ for each } \alpha < \kappa \right\}$$

is dense in $(G^*)^\omega$;

(b) *if $\kappa < \mathfrak{c}$, the conclusion in* (a) *remains valid under* MA.

Now we are in position to prove a version of Theorem 3.1 adjusted for Abelian groups of prime order.

Theorem 4.3. *Let (G,τ) be an Abelian topological group of a prime order p satisfying $|G|=\mathfrak{c}=|\tau|$. Then under MA, there exists a group topology $\mathcal{T}$ on G with the following properties*:

(a) *the topologies τ and $\mathcal{T}$ are independent*;
(b) *the group $(G,\mathcal{T})$ is countably compact and does not contain non-trivial convergent sequences*;
(c) *the group $(G,\mathcal{T})$ is strongly zero-dimensional.*

Proof. To avoid repetitions and shorten the argument, we follow notation in the proof of Theorem 3.1. Put $K=\mathbb{Z}(p)^\omega$. We will define an algebraic monomorphism $h:G\to K^{\mathfrak{c}}$ satisfying conditions (1) and (2) of Lemma 3.2. To this end, it suffices to guarantee the following properties of $h(G)$:

(A) $h(G)$ is an HFD subgroup of $K^{\mathfrak{c}}$, i.e., for every infinite subset S of $h(G)$ there exists $\alpha<\mathfrak{c}$ such that $\pi_{\mathfrak{c}\setminus\alpha}(S)$ is dense in $K^{\mathfrak{c}\setminus\alpha}$;
(B) $\pi_\alpha(h(G))=K^\alpha$ for each $\alpha<\mathfrak{c}$.

Then (A) and (B) will imply (a) and (b) of the theorem. In fact, (c) of the theorem also follows from (B) since the Stone–Čech compactification of a dense pseudocompact subset of an uncountable product of compact metrizable spaces is homeomorphic to the product space.

Represent G as the union of an increasing sequence $\{N_\alpha\colon\ \omega\leqslant\alpha<\mathfrak{c}\}$ of its subgroups N_α such that $|N_\omega|=\omega,\ |N_{\alpha+1}/N_\alpha|=\omega$ for each infinite $\alpha<\mathfrak{c}$, and $N_\alpha=\bigcup_{\omega\leqslant\beta<\alpha}N_\beta$ for each limit ordinal $\alpha<\mathfrak{c}$. In particular, $|N_\alpha|=|\alpha|$ if $\omega\leqslant\alpha<\mathfrak{c}$. Similarly to the proof of Theorem 3.1, there exists an enumeration $\{S_\mu\colon\ \omega\leqslant\mu<\mathfrak{c}\}$ of all countably infinite subsets of $G\setminus\{0_G\}$ such that $S_\mu\subseteq N_\mu$ for each infinite $\mu<\mathfrak{c}$. Let also $\{b_\mu\colon\ \mu<\mathfrak{c}\}$ be an enumeration of the set

$$\Sigma=\bigl\{x\in K^{\mathfrak{c}}\colon\ \exists\alpha<\mathfrak{c}\text{ such that }x(\beta)=0\text{ whenever }\alpha\leqslant\beta<\mathfrak{c}\bigr\}$$

such that every $x\in\Sigma$ occurs $\mathfrak{c}$ times in this enumeration. So, it remains to construct a family $\mathcal{H}=\{h_{\alpha,\nu}\colon\ \alpha<\mathfrak{c},\ \omega\leqslant\nu<\mathfrak{c}\}$ satisfying the following conditions for all $\alpha,\beta,\mu,\nu<\mathfrak{c}$ with $\mu,\nu\geqslant\omega$:

(1) $h_{\alpha,\nu}:N_\nu\to K$ is a homomorphism and $h_{\alpha,\nu}$ extends $h_{\alpha,\mu}$ if $\omega\leqslant\mu<\nu$;
(2) the image $(\Delta_{\mu\leqslant\gamma<\alpha}h_{\gamma,\nu})(S_\mu)$ is dense in $K^{\alpha\setminus\mu}$ whenever $\mu<\alpha$ and $\omega\leqslant\mu<\nu$;
(3) there exists a point $x\in\mathbb{N}_{\nu+1}\setminus N_\nu$ such that $h_{\gamma,\nu+1}(x)=b_\nu(\gamma)$ for each $\gamma<\nu$;
(4) if $\alpha\in L$ and $\varphi(\alpha)=(\mu,\delta)$, then there exists a point $y_\alpha\in F_\mu$ such that $\xi(y_\alpha)=\alpha$ and $h_{\gamma,\alpha}(y_\alpha)\in p_\gamma(O_\delta)$ for each $\gamma<\alpha$;
(5) if $\omega\leqslant\beta<\mathfrak{c}$ and $\xi=\min\{\xi(x)\colon\ x\in S_\beta\}$, then there exists a point $z\in S_\beta\cap N_\xi$ such that $h_{\beta,\beta+1}(z)\neq 0_K$.

Here $\{F_\beta\colon\ \beta<\mathfrak{c}\}$ is the family of all closed subsets of (G,τ) which have cardinality $\mathfrak{c}$, and $\{O_\delta\colon\ \delta<\mathfrak{c}\}$ is the family of all canonical clopen sets in $K^{\mathfrak{c}}$. We can assume that $\operatorname{coord}(O_\delta)\subseteq\delta$ for each $\delta<\mathfrak{c}$. The functions $\varphi:L\to\mathfrak{c}\times\mathfrak{c}$ and $\xi:G\to\mathfrak{c}\setminus\omega$ in (4) and (5) have the same meaning as in the proof of Theorem 3.1, where $L = \{\omega\} \cup \{\beta + 1\colon\ \omega \leqslant \beta < \mathfrak{c}\}$. Note that the above conditions (1)–(4) are exactly the same as in the proof of Theorem 3.1 (the only difference is that the group $\mathbb{T}$ is replaced by $K = \mathbb{Z}(p)^{\omega}$).

Suppose that the family $\mathcal{H}$ satisfying (1)–(5) has been constructed. Then for every $\alpha < \mathfrak{c}$, there exists a homomorphism $h_\alpha : G \to K$ whose restriction to N_ν coincides with $h_{\alpha,\nu}$ for each infinite $\nu < \mathfrak{c}$. Denote by h the diagonal product of the family $\{h_\alpha\colon\ \alpha < \mathfrak{c}\}$. The argument in the proof of Theorem 3.1 implies that the homomorphism $h : G \to K^{\mathfrak{c}}$ satisfies (1) and (2) of Lemma 3.2. However, the verification of the injectivity of h is now different, it depends on the new condition (5). Indeed, let $z \in G$ be an arbitrary non-zero element. Since $|N_\nu| < \mathfrak{c}$ for each $\nu < \mathfrak{c}$, there exists a countably infinite subset S of G such that $z \in S$ and $\xi(z) < \xi(x)$ for each $x \in S \setminus \{z\}$. Clearly, $S = S_\beta$ for some β with $\omega \leqslant \beta < \mathfrak{c}$. Then $h_{\beta,\beta+1}(z) \neq 0_K$ by (5), and hence $z \notin \ker(h)$.

Finally, we show that the required family $\mathcal{H}$ exists under MA. Let $\{h_{k,\omega}\colon\ k \in \omega\}$ be an arbitrary family of homomorphisms of N_ω to K. As in the proof of Theorem 3.1, there is no need to verify that this family satisfies (1)–(5). Suppose that for an infinite ordinal $\beta < \mathfrak{c}$, we have defined the family $\{h_{\gamma,\nu}\colon\ \gamma < \beta,\ \omega \leqslant \nu \leqslant \beta\}$ satisfying (1)–(5) (we consider only a non-limit step of the construction). Put $\alpha = \beta + 1$. We have to extend the homomorphisms $h_{\gamma,\beta}$ (with $\gamma < \beta$) over N_α, thus obtaining the homomorphisms $h_{\gamma,\alpha}$, and then we shall construct a homomorphism $h_{\beta,\alpha} : N_\alpha \to K$.

As in the proof of Theorem 3.1, we start to extend the homomorphisms $h_{\gamma,\beta}$ over N_α. There exists a subgroup P of N_α such that $N_\alpha = N_\beta \oplus P$. Then $P \cong N_\alpha / N_\beta$, so that P is infinite. Let $\varphi(\alpha) = (\mu, \delta)$. There exists a point $x \in F_\mu$ such that $\xi(x) = \alpha$ or, equivalently, $x \in N_\alpha \setminus N_\beta$. Then $x = x_0 + x_1$, where $x_0 \in N_\beta$ and $x_1 \in P$. For every $\gamma < \beta$, denote by $O_{\delta,\gamma}$ the projection of O_δ to the factor $K_\gamma = K$. Choose a point $y_\alpha \in P \setminus \langle x_1 \rangle$. Then the sum $N_\beta + \langle x \rangle + \langle y_\alpha \rangle$ is direct in N_α. Consequently, given any $a, b \in K$, every homomorphism $f : N_\beta \to K$ extends to a homomorphism $g : N_\alpha \to K$ such that $g(x) = a$ and $g(y_\alpha) = b$. For every $\gamma < \beta$, this observation enables us to define a homomorphism $h_{\gamma,\alpha} : N_\alpha \to K$ extending $h_{\gamma,\beta}$ and satisfying

$$h_{\gamma,\alpha}(x) \in O_{\delta,\gamma} \quad \text{and} \quad h_{\gamma,\alpha}(y) = b_\beta(\gamma).$$

This implies (1), (3) and (4) (for all $\gamma < \beta$) at the step α of our construction.

It remains to define a homomorphism $h_{\beta,\alpha} : N_\alpha \to K$ satisfying (2), (4) and (5). Choose a point $z \in S_\beta$ such that $\xi(z) = \min\{\xi(x)\colon\ x \in S_\beta\}$. Similarly to the proof of Theorem 3.1, denote by $f_{\mu,\beta}$ the diagonal product of homomorphisms $h_{\gamma,\beta}$ with $\mu \leqslant \gamma < \beta$ and consider the family

$$\mathcal{K}_\beta = \bigl\{S_\mu \cap f_{\mu,\beta}^{-1}\bigl(U \cap f_{\mu,\beta}(S_\mu)\bigr)\colon\ U \in \mathcal{U}_{\mu,\beta},\ \omega \leqslant \mu < \beta\bigr\} \cup \{S_\beta\},$$

where $U_{\mu,\beta}$ is the family of all canonical clopen sets in $K^{\beta\setminus\mu}$. Since $|\mathcal{U}_{\mu,\beta}| \leqslant |\beta|$ for each $\mu < \beta$, the cardinality of $\mathcal{K}_\beta$ does not exceed $|\beta|$. Note that by (2) (with β instead of α), all elements of $\mathcal{K}_\beta$ are infinite. Therefore, by Lemma 4.2(b), we can find a homomorphism $h_{\beta,\alpha} : N_\alpha \to K$ such that $h_{\beta,\alpha}(S)$ is dense in K for each $S \in \mathcal{K}_\beta$. In addition, we can choose $h_{\beta,\alpha}$ to satisfy $h_{\beta,\alpha}(y_\alpha) \in O_{\delta,\beta}$ and $h_{\beta,\alpha}(z) \neq 0_K$. Indeed, consider the group $\mathrm{Hom}(G, K)$

of all homomorphisms of G to K which can be identified with the compact topological group $(G^*)^\omega$. From Lemma 4.2(b) it follows that the set

$$H_\beta = \{f \in \mathrm{Hom}(G,K)\colon\ f(S) \text{ is dense in } K \text{ for each } S \in \mathcal{K}_\beta\}$$

is dense in $\mathrm{Hom}(G,K)$. The non-empty set

$$W = \{f \in \mathrm{Hom}(G,K)\colon\ f(y_\alpha) \in O_{\delta,\beta},\ f(z) \neq 0_K\}$$

is open in $\mathrm{Hom}(G,K)$, so $H_\beta \cap W \neq \emptyset$. It remains to choose $h_{\beta,\alpha} \in H_\beta \cap W$. This choice of $h_{\beta,\alpha}$ ensures the validity of (2) for $\alpha = \beta + 1$ (the argument is the same as in the proof of Theorem 3.1). Clearly, (4) and (5) also hold. This finishes our construction of the family $\mathcal{H} = \{h_{\alpha,\nu}\colon\ \alpha < \mathfrak{c},\ \omega \leqslant \nu < \mathfrak{c}\}$ satisfying (1)–(5). □

Corollary 4.4. *Under* MA*, every Abelian topological group G of prime order satisfying $w(G) < |G| = \mathfrak{c}$ admits an independent group topology.*

The proof of the following result is similar to that of Lemma 3.12, so we omit it.

Lemma 4.5. *Let S be an infinite subset of a countable Abelian group of prime order p. Then there exists a monomorphism $f : G \to \mathbb{Z}(p)^\omega$ such that $f(S)$ is dense in $\mathbb{Z}(p)^\omega$.*

Now we prove a counterpart of Theorem 3.13 for Abelian groups of prime order.

Proposition 4.6. *All proper unconditionally closed subsets of an Abelian group G of prime order p are finite.*

Proof. Since G is a vector space over the field $\mathbb{Z}(p)$, it is isomorphic to the direct sum of copies of $\mathbb{Z}(p)$. Let S be a proper infinite subset of G. If $|G| = \omega$, we apply Lemma 4.5 to conclude that S is not unconditionally closed in G. Suppose, therefore, that $|G| > \omega$.

There exists a countable subgroup P of G such that $S' = S \cap P$ is a proper infinite subset of P. Every subgroup of G is a direct summand in G [31], so there exists a subgroup Q of G such that $G = P \oplus Q$. Again, apply Lemma 4.5 to define a monomorphism $h : P \to \mathbb{Z}(p)^\omega$ such that $h(S')$ is dense in $\mathbb{Z}(p)^\omega$ and consider the Hausdorff group topology $\mathcal{T}$ on P associated with h. Clearly, S' is a proper dense subset of $(P,\mathcal{T})$. Denote by $(G,\mathcal{T}^*)$ the direct sum of topological groups $(P,\mathcal{T})$ and $(Q,\mathcal{T}_d)$, where $\mathcal{T}_d$ is the discrete topology on Q. The topological group $(G,\mathcal{T}^*)$ is Hausdorff and P is closed in it. If S were unconditionally closed in G, the intersection $S' = S \cap P$ would be closed in $(P,\mathcal{T})$, a contradiction. Thus, G has no proper infinite unconditionally closed subsets. □

5. General case

Here we join the results obtained separately for Abelian almost torsion-free groups and groups of a prime order in Sections 3 and 4. First, we characterize the Abelian groups which have all proper unconditionally closed subsets finite. Note that our proof of the following theorem does not depend on extra set-theoretic assumptions.

Theorem 5.1. *For an Abelian group G, the following are equivalent*:

(a) *all proper unconditionally closed subsets of G are finite*;
(b) *all proper unconditionally closed subgroups of G are finite*;
(c) *G is either an almost torsion-free group or has a prime order.*

Proof. It is clear that (a) implies (b). We will show that (b) implies (c) and (a) follows from (c). Note that for every $p \in \mathbb{P}$, the subgroup $G[p] = \{x \in G\colon\ px = 0_g\}$ of G is algebraically (hence unconditionally) closed in G.

Suppose that all proper unconditionally closed subgroups of G are finite. If $G[p]$ is infinite for some $p \in \mathbb{P}$, then $G = G[p]$, i.e., p is the order of G. Otherwise the groups $G[p]$ are finite, and hence $r_p(G) < \infty$ for each $p \in \mathbb{P}$. In other words, the group G is almost torsion-free. This proves that (b) implies (c).

Suppose now that (c) holds. If the group G is almost torsion-free, then an application of Theorem 3.13 finishes the proof. Otherwise G is of a prime order p, and hence Proposition 4.6 applies. □

The combination of Theorems 5.1, 3.1 and 4.3 implies our main result about independent group topologies:

Theorem 5.2. *Let G be an Abelian group of size $\mathfrak{c}$ without non-trivial unconditionally closed subgroups, and let τ be a Hausdorff group topology on G with $|\tau| = \mathfrak{c}$. Then under MA, there exists a Hausdorff group topology $\mathcal{T}$ on G with the following properties*:

(a) *the topologies τ and $\mathcal{T}$ are independent*;
(b) *the group $(G, \mathcal{T})$ is countably compact and does not contain non-trivial convergent sequences.*

The following result is probably known in the topological group folklore. It will be used to characterize "small" Abelian groups that admit independent group topologies (see Corollary 5.4). We give its proof here for the reader's convenience.

Lemma 5.3. *Every Abelian group G of size not greater than $\mathfrak{c}$ admits a second countable precompact Hausdorff group topology.*

Proof. Every infinite Abelian group embeds as a subgroup into a divisible group of the same cardinality [31, 4.1.6], so we can assume that G is divisible itself. Therefore, G is isomorphic to a direct sum of at most $\mathfrak{c}$ copies of the group $\mathbb{Q}$ or the quasicyclic groups $\mathbb{Z}_{p^\infty}$ with $p \in \mathbb{P}$ [31, 4.1.5]. Let $G_0 = \bigoplus_{\alpha<\mathfrak{c}} \mathbb{Q}_\alpha$ and $G_p = \bigoplus_{\alpha<\mathfrak{c}} (\mathbb{Z}_{p^\infty})_\alpha,\ p \in \mathbb{P}$, where $\mathbb{Q}_\alpha = \mathbb{Q}$ and $(\mathbb{Z}_{p^\infty})_\alpha = \mathbb{Z}_{p^\infty}$ for each $\alpha < \mathfrak{c}$. From Lemma 3.11 (with $N = \{0\}$) it follows that $\mathbb{T}$ contains a subgroup isomorphic to G_0. In addition, for every prime p, the group $\mathbb{T}^\omega$ contains the subgroup $\mathbb{Z}(p)^\omega$ which is algebraically isomorphic to the direct sum of $\mathfrak{c}$ copies of the group $\mathbb{Z}(p)$. Since $\mathbb{T}^\omega$ is divisible, this immediately implies that $\mathbb{T}^\omega$ contains a subgroup isomorphic with G_p.

Therefore, the group $G_0 \oplus (\bigoplus_{p\in\mathbb{P}} G_p)$ contains G and is algebraically isomorphic to a subgroup of $(T^\omega)^\omega \cong \mathbb{T}^\omega$, so the usual product topology of $\mathbb{T}^\omega$ induces on G a second countable precompact Hausdorff group topology. □

Theorem 5.2 and Lemma 5.3 together imply a characterization of the Abelian groups of size $\mathfrak{c}$ that admit independent group topologies.

Corollary 5.4. *Under* MA, *an Abelian group G of size $\mathfrak{c}$ admits a pair of independent group topologies iff all proper unconditionally closed subgroups of G are finite.*

This result, however, leaves open the problem of whether independent group topologies exist in ZFC (see Problems 6.3 and 6.4).

Finally, we present a sufficient condition for Abelian groups of size $\mathfrak{c}$ to admit a countably compact Hausdorff group topology. A complete characterization of the algebraic structure of countably compact Abelian groups of size $\mathfrak{c}$ will appear in [12].

Corollary 5.5. *Suppose that MA holds. Then every Abelian group of size $\mathfrak{c}$ without non-trivial unconditionally closed subgroups admits a countably compact group topology.*

Proof. By Lemma 5.3, an Abelian group G with $|G| = \mathfrak{c}$ admits a second countable group topology τ. Apply Theorem 5.2 to find a group topology $\mathcal{T}$ on G independent of τ. By Proposition 2.4, the group $(G, \mathcal{T})$ is countably compact. □

6. Open problems

Here we collect several open problems concerning independent and countably compact group topologies. It seems to us that finding solutions to Problems 6.3, 6.4 and 6.6 may present significant difficulties.

Problem 6.1. Let τ be the usual topology on $\mathbb{R}$. Does there exist (under MA or CH) a Hausdorff group topology $\mathcal{T}$ on $\mathbb{R}$ such that the topologies τ^2 and $\mathcal{T}^2$ on $\mathbb{R}^2$ are independent? What about the independence of all finite powers of τ and $\mathcal{T}$? What is the answer if one takes a second countable Hausdorff group topology τ on an arbitrary almost torsion-free Abelian group G with $|G| = \mathfrak{c}$?

Problem 6.2. Do there exist two independent locally compact Hausdorff group topologies?

The answer to the above problem is probably "no": locally compact groups contain a lot of convergent sequences (compare with Proposition 2.4).

Problem 6.3. Is it true in ZFC that every Abelian torsion-free group of cardinality $\mathfrak{c}$ admits two independent group topologies?

Note that under MA, every Abelian torsion-free group of cardinality $\mathfrak{c}$ admits a pair of independent group topologies (see Corollary 5.4 and Theorem 5.1). This is why we ask for a ZFC solution to Problem 6.3. The next problem seems to be especially difficult if $\omega < |G| < \mathfrak{c}$ or $|G| > \mathfrak{c}$. Consistent results will probably come first.

Problem 6.4. Does every uncountable Abelian group without proper infinite unconditionally closed subsets admit two independent Hausdorff group topologies?

The problem below arises as an attempt to extend Theorem 3.10 to a wider class of Abelian groups.

Problem 6.5. Is it true that every infinite subset of an almost torsion-free Abelian group G with $|G| \leqslant 2^{\mathfrak{c}}$ is potentially dense in G?

The last two problems outline the limits of our knowledge about countably compact topological groups (see also [5, Question 3.9.3] and [10, Question 5.16]).

Problem 6.6. Is it true in ZFC that every Abelian torsion-free group of cardinality $\mathfrak{c}$ admits a countably compact group topology?

Problem 6.7. Does the free Abelian group of cardinality $\mathfrak{c}^+$ or $2^{\mathfrak{c}}$ admit a countably compact group topology? More generally, does every Abelian torsion-free group of cardinality $\mathfrak{c}^+$ or $2^{\mathfrak{c}}$ admit a countably compact group topology?

Acknowledgements

The authors are grateful to Dikran Dikranjan for helpful discussions that resulted in a substantial improvement of the article, and to the referee for a careful reading of the manuscript and several useful remarks.

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